Falling spring

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Abstract

The paper describes two experiments concerning free fall of systems combining elastic and inertial mass. The described experiments represent two extreme situations: 1) elastic mass is negligible, 2) all the falling mass is elastic. Both experiments are very impressive and can be at certain level analysed using upper secondary school physics. The theoretical arguments are complemented by analysis of video recordings obtained using high-speed camera.

Introduction

The topic of the “falling spring” seems very trivial. Yet an experiment can be arranged in such a way, that it is not only very appealing, but it can also be very well described using merely secondary school physics-level. This paper explores two variants of these experiments. The first one, which is physically easier, contains two falling object vertically connected by a spring of negligible mass. In the second variant all of the falling mass is part of the spring – the fall of a freely hanging toy – the so-called “slinky”. The results of the latter experiment are very surprising and can be interestingly commented on using basic laws of physics, despite the complexity of its theoretical description.

Two objects connected by a spring

In this experiment, we let two bodies of equal mass connected by a spring fall freely from an initial position where one body is held in the hand and the other is hanging freely on the spring, see figure 1(a). The motion itself is very fast, so a high-speed camera is necessary for observation or measurement. The very beginning of the fall can be quite surprising - the lower body practically does not fall at all for a relatively long time - yet this can be explained by a very simple reasoning (see [1] as well).

The force analysis is shown in figure 1(b). The gravitational forces acting on both bodies are vertically downward (red arrows). Considering the equal mass of the bodies, the magnitudes of the forces are identical. Before the fall commences, both bodies are stationary, thus the resulting forces applied to each individual body must be equal to zero. Consequently, the magnitude of the spring tensile force acting on the lower body must be the same as the gravitational force, and according to the law of action and reaction, the same magnitude of the tensile force acts on the upper body, only in the opposite direction (blue arrows). To achieve force equilibrium on the upper body, a pull is required from the suspension (hand), which is not plotted in figure 1(b), so the figure shows the state immediately after the suspension is released.

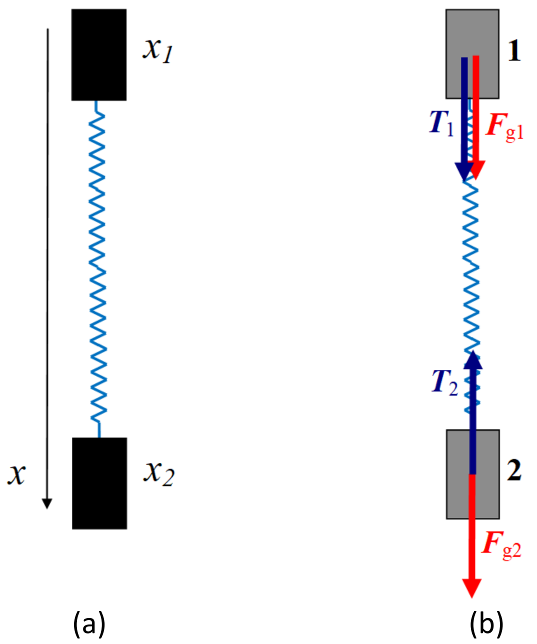

Fig. 1. Two bodies connected by a massless spring (a) and force analysis (b).

The force analysis immediately gives that the resultant force at the moment after the suspension is released is zero on the lower body, so the body does not accelerate at all. On the contrary, the upper body is subjected to twice the force of gravity and therefore "falls" with an acceleration of 2*g*.

The next stages of this experiment can be described theoretically using only upper secondary school physics and mathematics.

Let us first solve the problem in a centre-of-mass reference frame and let us align the system horizontally for easier visualization, see figure 2. In the centre-of-mass frame, the centre of mass cannot move, so the motions of the two bodies relative to the centre of mass must be symmetrical. This also means that the situation does not change if we firmly fix the centre of mass to a point in space, see figure 3. In that case, we can solve the left and right symmetric parts separately and obtain a pretty standard upper secondary school problem of oscillations of a body on a fixed spring, see figure 4. However, it needs to be considered that a spring of half the length has twice the stiffness.

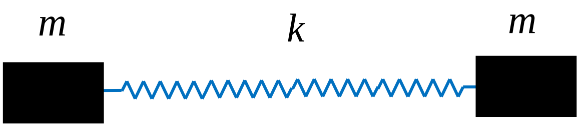

Fig. 2. Two bodies connected via a spring in a centre-of-mass reference frame.

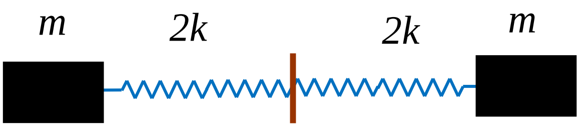

Fig. 3. Two bodies connected via a spring in a centre-of-mass reference frame, the position of the centre of mass is fixed.

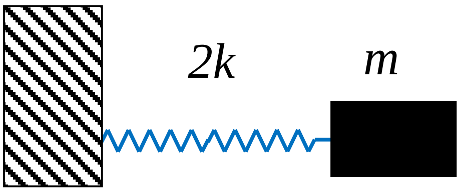

Fig. 4. A body on a spring, a standard secondary school task.

Due to the initial conditions

$$x(t=0)=x_o \quad (1)$$

$$v(t=0)=0, \quad (2)$$

the solution of the following form is obtained:

$$x(t)=x_o\cos\omega t, \quad (3)$$

for

$$\omega=\sqrt{\frac{2k}{m}}. \quad (4)$$

It needs to be noted, that for this experiment the solution can only be used for short times compared to the theoretical oscillation period.

Now let us apply the solution easily obtained in the centre-of-mass frame to the original problem. The static spring extension determines the amplitude of the displacement

$$x_o=\frac{mg}{2k} \quad (5)$$

and the transition from the centre-of-mass frame to the laboratory frame is performed by adding the free fall with acceleration *g* to the time evolution of the position. We get:

$$x_1(t)=-\frac{mg}{2k}\cos\omega t+\frac{1}{2}gt^2, \quad (6)$$

$$x_2(t)=\frac{mg}{2k}\cos\omega t+\frac{1}{2}gt^2. \quad (7)$$

The coordinates x_1 and x_2 determine the position of both bodies relative to the equilibrium position in which the spring connecting the two bodies would be unstressed. The negative sign of the position of the upper body is given by the choice of the orientation of the x-axis, see figure 1(a).

Short-time approximations

The previously stated solution can be complemented by an illustration of approximation methods, which are used very often in physics.

Let us approximate above mentioned solution for short times, i.e. for times much shorter than is the period of the oscillations

$$\omega\, t \ll 1. \quad (8)$$

Hence, the cosine function can be approximated as

$$\cos \omega t \cong 1 - \frac{1}{2}(\omega t)^2. \quad (9)$$

In this approximation equations (6) and (7) can be written as

$$x_1(t) = -\frac{mg}{2k} + \frac{1}{2} 2gt^2, \quad (10)$$

$$x_2(t) = \frac{mg}{2k}. \quad (11)$$

So, the calculation in the small-time approximation showed that at the beginning of the fall the upper body is indeed falling with an acceleration of $2g$, while the lower body is not accelerating at all. The same conclusion was drawn from the initial simple force analysis, but the approximate calculation gave information not only about the initial moment but also about the time dependence in the initial part of the fall.

Experimental verification

Experimental verification was carried out with a steel spring of mass m_p = 0.1 kg, stiffness k = 7.7 N/m and with two one-kilogram weights. Thus, the mass of the spring was approximately equal to 5 % of the mass of the whole system and it was therefore possible to apply the above-described model with a massless spring. The system was suspended on a thin thread and was released by burning the thread. In addition, a thicker string was attached to the upper body, which was used by the experimenter to catch the falling body after some time, so the spring would not be damaged by the impact of the heavy body.

The fall was recorded by a high-speed camera with a frame rate of 1000 Hz. The video was processed using Capstone.

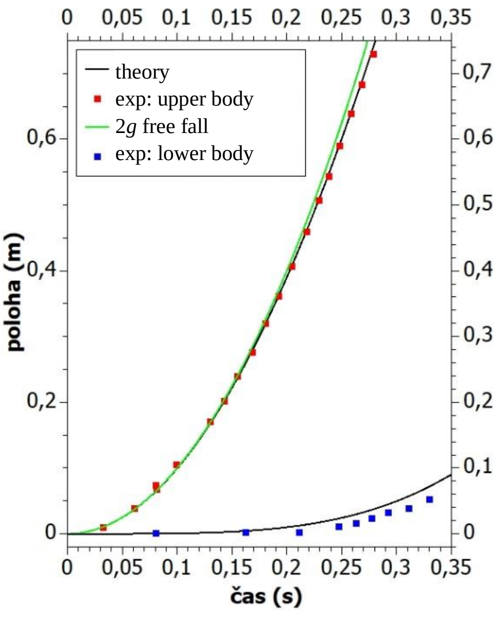

Fig. 5. Time dependency (*horizontal axis*) of the position (*vertical axis*) during the fall of two bodies.

The result is shown in figure 5. The initial positions of both bodies are placed to the origin of the coordinate system. For the upper body, the experimental data are in very good agreement with the theoretical calculation according to equation (6). The experimental data for the lower body are slightly delayed compared to the theory (equation (7)). This effect is due to the finite wave propagation speed in the spring and will be further discussed later.

The green curve shows the free fall of the body with a $2g$ acceleration. We can see that in the first 0.15 s or so of the fall the upper body is truly falling with an acceleration of 2g, but in the latter stages of the fall the acceleration starts to noticeably decrease.

„Slinky“

Slinky is a spring with a larger number of turns and very small stiffness, so that it sags significantly under its own weight when hanging freely, see figure 6. Slinky is commonly sold as a toy that exhibits interesting effects, see for example [2].

Fig. 6. Slinky.

The free fall of a vertically hanging slinky is a very interesting and surprising phenomenon that begs for a theoretical description. It can be said that the problem is currently theoretically solved [3]. The analytical solution is well beyond the level of secondary school physics; nonetheless, interesting conclusions can be drawn using the basic laws of mechanics even in secondary school.

The video is accessible, for example, at [4]. Similar videos can easily be found on YouTube by inputting the keyword "Falling slinky".

Two facts about the fall are particularly surprising.

1) The spring contracts in such way that the upper coils gradually overlap, and this growing bundle travels the entire length of the stationary spring.
2) Throughout the contraction of the spring, the lower end of the spring remains motionless.

Both effects can be explained by the remarkable property of the longitudinal wave propagation speed in the spring. From the theoretical analysis stems that the longitudinal wave propagates at the speed at which the wave travels through the same mass of the spring in the same time period. If we define the so-called "mass velocity" (with the unit kg/s), then the following applies [5]

$$v_m = \sqrt{k \cdot m_p} = \text{konst}, \tag{12}$$

where m_p je the mass of the whole spring. The constancy of the "mass velocity" means that the time of flight of the wave from one turn to another is constant and independent of the instantaneous extension of the spring. During static sag, the density of the turns increases downwards, so a wave with a constant mass velocity slows down as it passes through the spring, causing compression of the falling turns. The part of the spring below the compressed bundle has not yet received the information that the upper end of the spring has been released, so the spring remains in its pre-release state. Thus, the lower end of the spring remains motionless throughout the wave passage through the spring.

Weighted spring

Immobility of the lower end of the spring throughout the contraction of the spring is a very surprising phenomenon. What would happen if we were to load the lower end with some additional weight? At first glance, it would seem that the gravitational force on this weight would cause the lower end of the spring to accelerate. However, it can be very easily reasoned that even in this case the lower end of the spring will remain stationary until the compressed pack of turns reaches it. Let us consider an arbitrary turn in the middle part of the spring. When the spring hangs freely, this coil is deformed by the gravitational force of the mass under the turn. It does not matter whether the mass below consists of the other turns of the spring or of the added weight. Consequently, the longitudinal wave has no information about the distribution of the mass in the lower part of the spring and its contraction proceeds in the same way.

An intriguing consequence of the constancy of the mass velocity is the fact that regardless of the static extension of the spring, i.e. regardless of the mass of the weight suspended under the spring, the flight time through the entire length of the spring is always the same. The time of flight over the entire length of the spring is calculated as (analogy of the relation $t = s/v$)

$$t_p = \frac{m_p}{\sqrt{k \cdot m_p}} = \sqrt{\frac{m_p}{k}} \tag{13}$$

and this time depends exclusively on the stiffness of the spring and its total weight and does not depend on the current spring extension.

The equation (12) and the consequent (13) cannot be deduced in the secondary school physics framework and must be stated as a fact. However, it can be very easily experimentally verified. Fasten the lower end of the spring to the ground via an electronic force gauge. Fasten another force gauge to the upper end and stretch the spring vertically by hand. Generate a longitudinal wave that propagates downwards through the spring by swinging the arm vertically. Longitudinal wave oscillations are registered by both force gauges and from the signal delay in the lower force gauge the propagation time of the wave along the entire length of the spring can be determined. Repeat the experiment for different static spring tensions. An example of such measurement is shown in figure 7. For the measurements, the steel spring from the first experiment was used, which was successively stretched with a force of 5 N (upper pair of curves) and a force of 10 N (lower pair). In both cases, the pulse delay at the lower end of the spring (red curve) compared to the upper end of the spring (black curve) is identical within the experimental errors. Thus, the travel time of the wave along the spring indeed does not depend on its tension.

Substituting the parameters of the used spring m_p = 0.1 kg, stiffness k = 7.7 N/m, yields

$$t_p = \sqrt{\frac{m_p}{k}} = \sqrt{\frac{0.1}{7.7}} = 0.114 \text{ s}, \qquad (14)$$

which is in good agreement with the measured values.

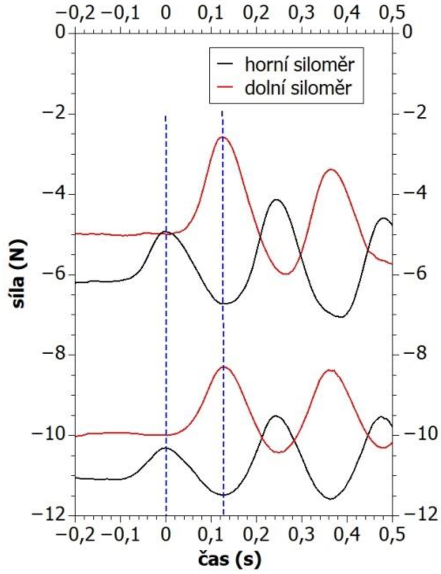

Fig. 7. Propagation of longitudinal waves in a slinky spring. (Horizontal axis – time; vertical axis – force; black curve – upper force gauge; red curve – lower force gauge.)

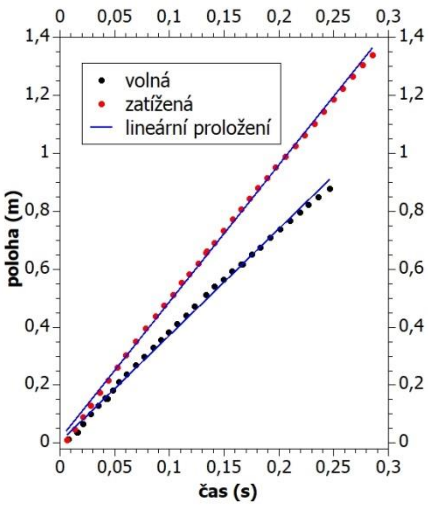

Fig. 8. Time dependency of the position of the upper end of the slinky. (Horizontal axis – time; vertical axis – position; black dots – a free slinky; red dots – a loaded slinky; blue line –a linear fit.)

Experiment

The fall of the slinky with different suspended weights was captured with a high-speed camera at 1000 Hz and assessed in Capstone. The results for two different spring loads are shown in figure 8. It can be seen that the upper end of the spring – the contracted bundle of turns – falls at a nearly constant rate in both cases (except for a brief moment immediately after the top body is released). A detailed comparison with linear fit even shows that the speed decreases during the fall. This can be explained qualitatively by the moving bundle successively impacting the lower turns causing them to accelerate, and this slows down its motion compared to the expected accelerating fall.

The contraction time of a differently loaded spring is not constant, as it should follow from equation (13). The spring with a higher load falls with higher speed but for a longer time than the spring with a lower load. This effect is probably due to air resistance, which is more significant for the faster falling spring.

Spring contraction and fall of the centre of mass

Provided that an arbitrary system of bodies falls freely in a homogeneous gravitational field, the centre of mass (centre of gravity) must move in a uniformly accelerated motion with acceleration g. However, the substantial part of the spring remains at rest for most of the contraction and the upper end falls at an approximately constant speed. Although this does not contradict the necessity of a free fall of the centre of gravity, the density of the turns increases along the downward direction and the mass of the approximately uniformly falling contracting bundle increases with increasing velocity, providing the desired uniformly accelerated fall of the centre of gravity.

Spring contraction and initial position of the centre of gravity

A very interesting conclusion stems from the fact that the contraction time of a spring does not depend on its deformation. Suppose that the contracted spring is much shorter than its initial length before relaxation, and thus we can neglect its contracted length. Let us

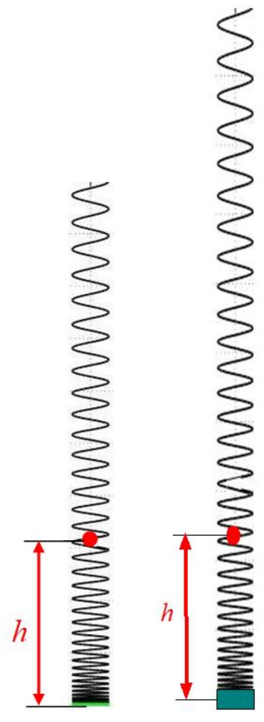

Fig. 9. Position of the centre of the gravity of freely suspended and loaded spring.

further assume that the vertical size of the added weight is also negligible relative to the initial length of the stretched spring. Under these simplifying assumptions, the entire spring (including any additional weight) contracts to a point at the lower end of the spring Simultaneously, the contraction time must be equal to the free fall time of the centre of gravity of the system. Due to the fact that the contraction time does not depend on the static sag, i.e. on the weight of the added weight, the centre of gravity of the system spring + weight must be in the same distance from the lower end of the spring, regardless of the weight of the extension, see figure 9.

Torsional wave

When closely observing the slow-motion recording of the falling spring, it can be noticed that the spring is not completely stationary under the falling bundle of turns, rather a torsional wave is propagating in the spring, which is related to the fact that the torsional rotation of the spring occurs at the same time as the spring is being stretched. This phenomenon can be easily demonstrated by a simple experiment: a load of a reasonable weight is suspended from a spring and placed on a mat at the beginning of the experiment. We mark a certain position on the circumference of the weight beforehand, so that we can observe the rotation well. We fully relax the spring. Then gradually lift the upper end of the spring until the suspended body rises above the mat. During the lifting, the weight visibly rotates, showing that the longitudinal stretching of the spring is accompanied by a torsional deformation.

A soft spring - the slinky - is stiffer against torsional rotation than against longitudinal extension, so the speed of propagation of the torsional deformation is higher than the speed of propagation of the longitudinal wave. However, the torsional wave does not cause vertical movement of the lower end of the spring, which remains stationary throughout the spring contraction.

Conclusion

The fall of the spring is an interesting and appealing problem. In the first variant - the elastic mass is negligible compared to the inelastic mass - the problem can be solved theoretically even within the framework of upper secondary school physics. The second variant - the falling slinky - is too difficult theoretically for upper secondary, yet this very attractive and surprising experiment can be commented on using the basic laws of mechanics. It can also be used as an illustration of the finite speed of propagation of a force interaction, or as an analogy for the tsunami phenomenon in which the phase velocity of a wave near the coast decreases, causing a catastrophic increase in amplitude. Similar to the fall of the slinky, the mass of contracted turns increases rapidly for the same reason.

Finally, the attractiveness and richness of the physical content of these experiments can be used as a subject for student projects.

References

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[3] Cross R. C., Wheatland, M. S. *Modeling a Falling Slinky*. Am. J. Phys. 80, 1051 (2012)

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[5] Edwards T. W., Hultsch R. A. *Mass Distribution and Frequencies of a Vertical Spring*, Am. J. Phys. 40, 445 (1972)