Square Bubbles

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Abstract

A bubble is a beautiful physical object. Blown from a small bubble blower, it forms the shape of a perfect sphere, flashing the colours of the rainbow and floating through the air as if gravity had little power over it. Each of these phenomena is underlined by interesting physics. In this post, we'll focus on the shape of bubbles, and not just the round ones.

Why are droplets and bubbles round?

Everyone knows round bubbles, because they can be observed with a mere eye, but photographs of falling drops prove that drops are round as well. Even droplets of water on leaves forms into the shape of a sphere. Why this shape?

A task: You have 1 dm^3 of modelling clay. Shape it into a cube, then into a sphere and then into two spheres. The volume is still the same – 1 dm^3. Compare surfaces of these objects.

Let's consider this a thought experiment rather than distribute a mound of clay to every student. Surely students can find out that the cube with a given volume has edges long 1 dm and a surface $S_{cu} = 6$ dm^2. Radius of a sphere needs to be determined from its volume and further used to calculate its surface which is $S_1 = 4.836$ dm^2. Overall surface of two created spheres is $S_2 = 6.093$ dm^2. Hence, a surface of one sphere is smaller than the surface of two spheres and smaller than the surface of a cube. We say that the liquid is forming a shape with minimal surface energy, therefore minimal surface, which is exactly the shape of one sphere.

The task can be pushed even further and derive a formula for S_N describing dependency of a total surface of N identical spheres. Secondary school student can derive that:

$$S_N = \sqrt[3]{36\pi V^2} \cdot \sqrt[3]{N}.$$

Thus, a thousand of the same spheres have surface 10 times greater than the surface of one sphere with equal volume.

Bubbles attached to a grate

Provided that a grate is submerged into e.g. water with dishwashing liquid, bubbles are formed on the grate creating interesting shapes – square bubbles. That can be surprising to an unknowing person (fig. 1) and even more surprisingly repeating this experiment yields the same shapes every time (as long as the bubble is attached to every edge). Seems like the grate remembers without a doubt the shape a bubble should create.

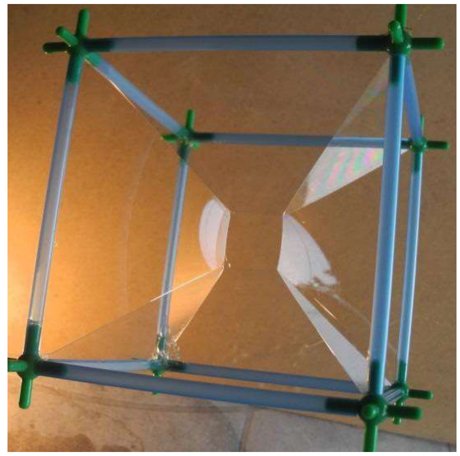

Fig. 1 – On the left is a bubble in a regular tetrahedron on the right a bubble in a cube.

Do square bubbles respect a minimal surface?

Looking for a minimum of a function is mathematical interesting task. Let us begin with a plane task (2D), rather than a task in 3D space.

A task: There are N points in a plane. Find a line connecting all the points and its length is minimal.

This is a mathematical task that belongs to a graph theory and leads to so-called **Steiner trees**.

A triangle

At first let's solve this task for $N = 3$. We can begin with a trial and error. In a free software GeoGebra (www.geogebra.org) a general triangle ABC can be modelled and a movable point X that can be connected to the vertices. The quantity d represents the studied length, i.e. the sum $d = |XA| + |XB| + |XC|$. GeoGebra allows for dynamic geometry, so it is possible to move point X around, while the momentary length d changes accordingly.

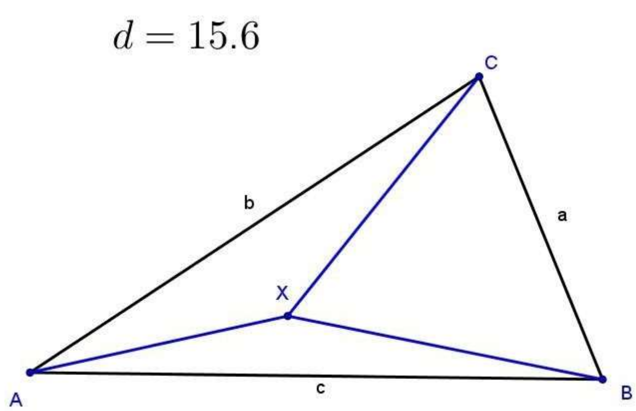

Fig. 2 – Looking for the shortest line connecting all the vertices of the triangle.

Placing point X into one of the vertices yields following values of d: **19,47**; **15,73**; **15,14**. We might think that d is minimal for the centroid (yield $d =$**14,61**) or for the orthocentre (yields $d =$**14,42**) or for the circumcentre (yields $d =$**15,32**) or for the incentre (yields $d =$**14,28**). Out of the points in question d is minimal for the incentre. However, is that an actual minimum?

This task of searching a point with a minimal distance to the vertices of a triangle was first formulated by P. Fermat (1601-1665) in a letter to E. Torricelli (1609-1647) who solved this task [1].

Construct an equilateral triangle ACD over the side AC of the triangle ABC and connect the vertex D with the vertex B. Repeat this process for the two remaining sides of ABC. It can be proven that line segments DB, EA and FC intersect in a single point X_0 called the first isogonic centre, or the so-called **Fermat-Torricelli point**.

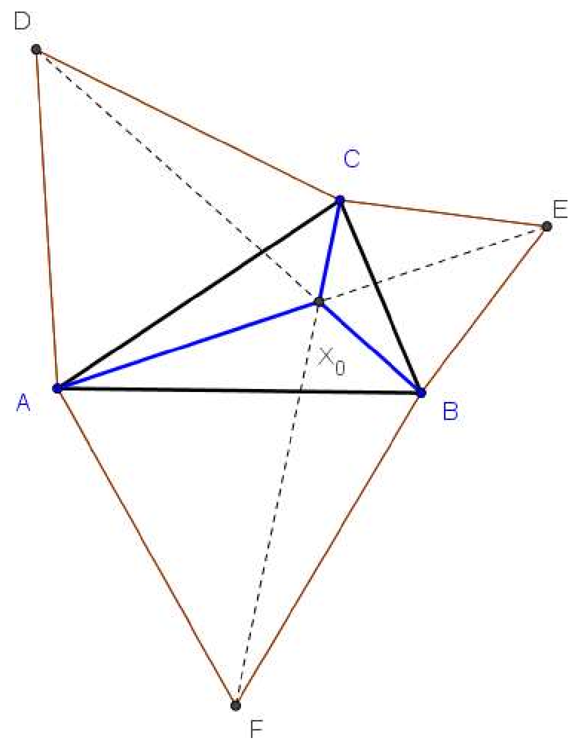

Fig. 3 – Construction of Fermat-Torricelli point X_0.

The website [1] contains three proofs that the distance is minimal and that this point has other interesting properties. E.g. that each side of the triangle (with interior angles less than 120°) can be seen with a viewing angle of 120° from the Fermat-Torricelli point.

A square

Let's solve this task for $N = 4$ and the quadrilateral in question will be a square with a side of 1. Connecting line consists of line segments, because it is a shortest connection of two point in a plane. Once again let's start with trial and error.

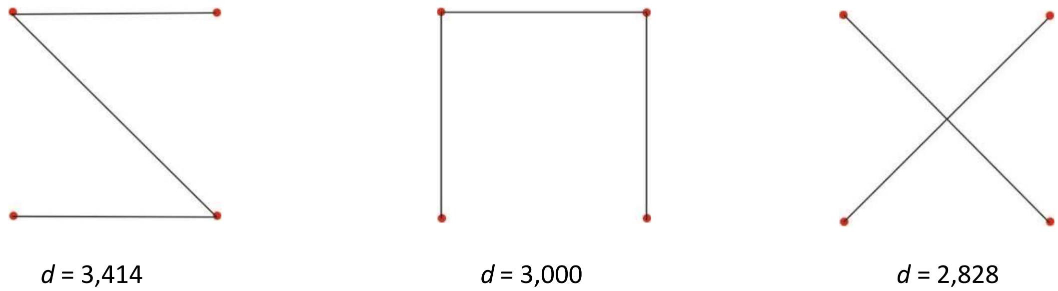

Fig. 4 – A handful of trees with the length of the connecting line.

It can be found, by using GeoGebra, that placing a point to the centre of the square results in minimal value of d. Yet even shorter connecting line exists.

A tree using two points inside of the square leads to the absolute minimum (fig. 5).

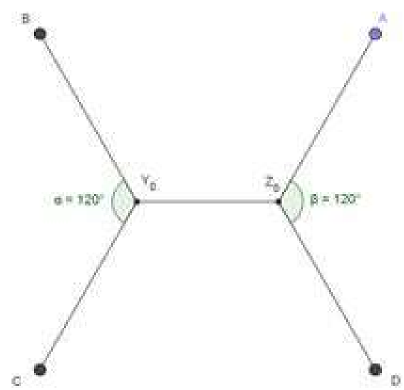

Fig. 5 – Shortest route connecting vertices of the square.

The line's length minimum can be found using derivation. It is not without interest that just as well as in the case of Fermat-Torricelli point 120° angles occur.

A regular tetrahedron

Let's add a dimension and solve minimal surface of a bubble attached to edges of a spatial grate.

A task: A bubble is attached to all 6 edges of the tetrahedron. Does a membrane with a minimal surface exist?

The bubble can choose the option to cover facets of the tetrahedron, hence creating 4 triangles. Because every membrane has two surfaces count them twice (though it is not important). Twice the surface of a tetrahedron with the length of the edge equal to 1 is 3.464 u^2. Keep this number in mind.

GeoGebra allows to conveniently model 3D bodies which can be freely rotated and viewed from different sides. Let's model (fig. 6) a shape of the bubble similar to the one observed in reality (on the left in fig. 1), with a movable point H on the segment GD.

Shape of the bubble in fig. 6 consists of 6 triangles (each with 2 surfaces). When placing the movable point H into the vertex D the surface is exactly 3.464 u^2, while placing it into the lower base of the tetrahedron changes the surface to 2.280 u^2, i.e. smaller value. However, the minimal value is 2.121 u^2.

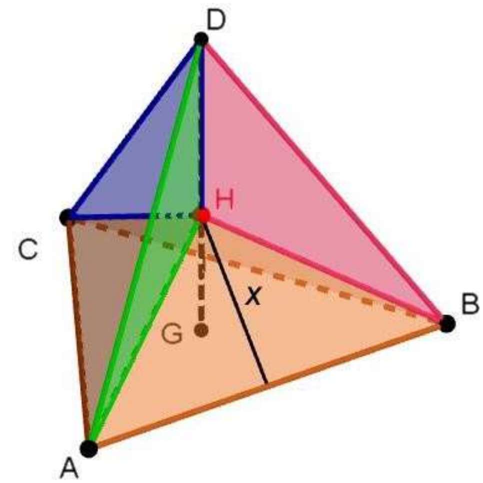

Fig. 6 – Searching for the bubbles shape in a regular tetrahedron.

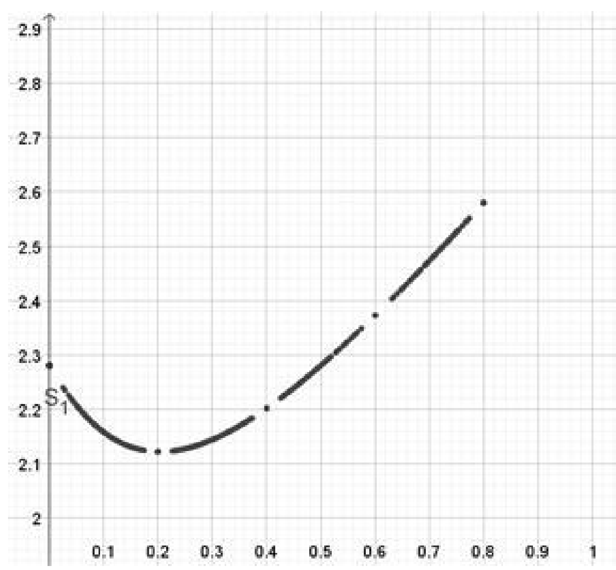

Fig. 7 - Dependence of the surface area of a bubble in a tetrahedron on the height of the point H above the base.

Let's assemble the function describing bubble's surface and find its minimum. It is convenient to select a height in a triangle e.g. ABH as a variable x. In this case twice the bubble's surface satisfies the equation:

$$S = 3x + \sqrt{2} - \sqrt{3x^2 - \frac{1}{4}}.$$

Minimal surface found using derivation occurs for $x = \frac{\sqrt{2}}{4}$. For this value the bubble consists of 6 congruent isosceles triangles with a vertex angle of 109.471°. Thus, we have verified the minimum surface principle for the tetrahedron.

A cube

Task: A bubble is attached to all the 12 edges of the cube. Does a membrane with a minimal surface exist?

Using GeoGebra a model of a cube with an edge 1 can be modelled with the bubble consisting of one square, four congruent isosceles triangles and eight congruent isosceles trapeziums as observed experimentally (on the right on fig. 1). Although the situation is spatial, we can see an analogy to fig. 5, which describes situation in a plane.

The point M is movable in such way that changing his position changes the length x of the square's edge. Plot the dependence of the bubbles surface on the variable x (fig. 9).

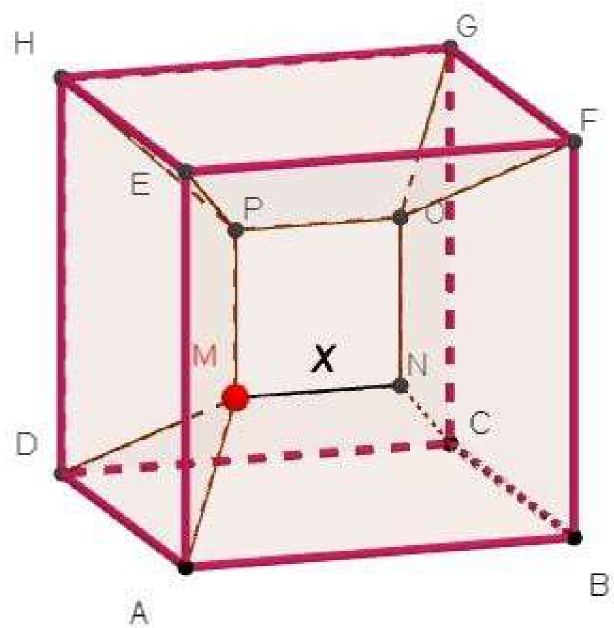

Fig. 8 – Searching for bubble's shape in the cube.

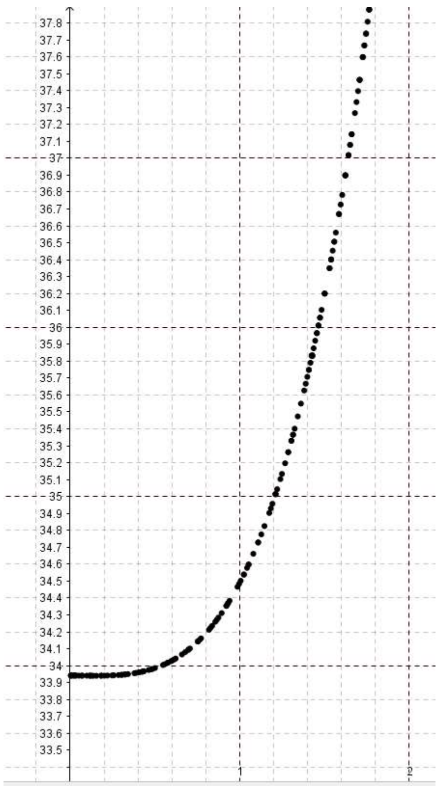

Fig. 9. Dependence of bubble's surface in a cube on variable x.

Whilst in fig. 7 an obvious minimum can be spotted a function in fig. 9 seems to reach minimum for $x = 0$, thus no square at all. Let's try to investigate the formula of the function describing dependence of the surface S on the x:

$$S = 2x^2 - 2\sqrt{2}x + (4x + 4)\sqrt{x^2 - 2x + 2} + 2\sqrt{2}$$

Locating function's extremes can be achieved by using GeoGebra function plotting tool, which can provide information about extremes (fig. 10).

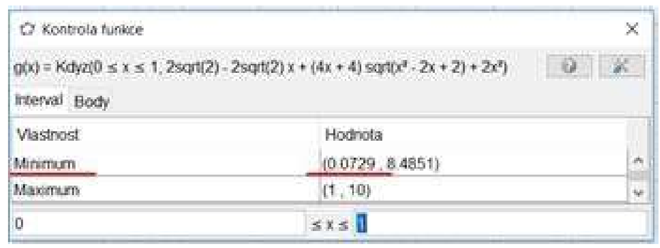

Fig. 10. Function's extremes in interval $\langle 0; 1\rangle$

So, a function's minimum doesn't occur in zero, but very close to zero for $x = 0.0729$. Hence, how big is the square in the cube we have used (fig.1; right side) supposed to be? Cube's edge is 105 mm long and the ratio of the square's side to the cube's edge for a minimal surface is equal to 0.0729. This results in $x = 7.7$ mm, which doesn't correspond to reality. Square's side is repeatedly more than two times longer. There must be some issue in our reasoning.

Plateau border

All previous thought assumed that the bubble consists of planar surfaces, that intersect in line segments. The reality is a bit different – the surfaces near intersections curve creating channel-like structures resembling an area between three touching circles – a Plateau border (fig. 11). Internet articles suggest that stable minimal surface is found using computer simulations and that it is probably not something to be studied in grammar school.

Perhaps, it may be summarised that in the case of the tetrahedron the bubble's geometry is such that the effect of Plateau border is negligible, yet in a cube it is not so. 2D (line connecting vertices) and 3D (edge union) analogy unfortunately didn't work out exactly well.

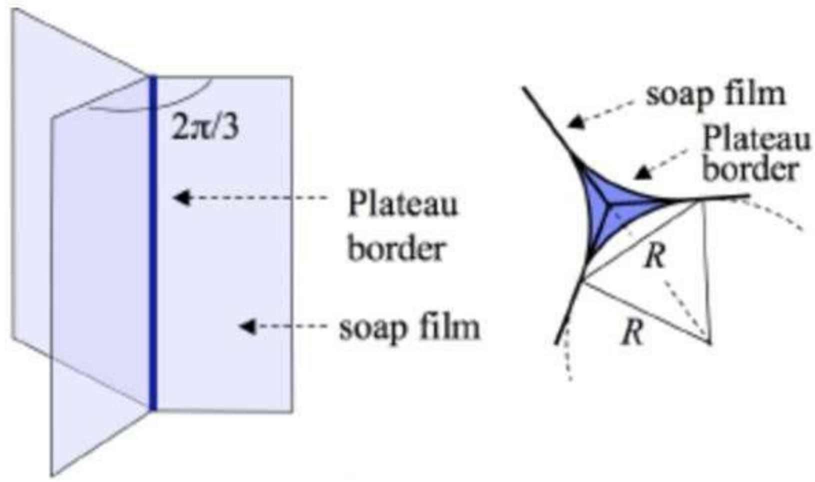

Fig. 11 - Plateau border at an intersect of three surfaces [2].

References

[1] https://en.wikipedia.org/wiki/Fermat_point
[2] https://www.researchgate.net/profile/Cyprien_Gay).