Paradox of Two Capacitors

JIŘÍ KOHOUT, PAVEL MASOPUST
Department of Physics, Faculty of Education, University of West Bohemia

Abstract

Currently, there is no problem to get quite cheap capacitors with the capacitance greater than 1 *F*. These components even without special measuring systems allow to carry out a lot of interesting experimental tasks, which bear relation to transient response and might help pupils to understand better this relatively difficult part of physics. The main aim of this paper is to theoretically and experimentally analyse the so-called paradox of two capacitors that refers to the apparent breaking of the energy conservation law during the transient response, when a capacitor is connected to another capacitor of the same capacitance. The paper also presents ideas for laboratory works which are executable in the teaching of electricity at upper secondary schools.

Introduction

The capacitor is one of the basic electrical components, and in the physics education at the upper secondary school a main focus is given to it at the end of the topic of electrostatics. This component is even the matter of some laboratory works carried out at schools, moreover the textbook of electricity and magnetism for general secondary schools describes the measurement of the capacitance of a capacitor by the help of alternating electric current, and websites of some schools (e.g. Jirásek general secondary school in Náchod - http://fyzika.gymnachod.cz/) offer manuals for the determination of capacitance from the discharge curve with the use of the Vernier system.

The Vernier system is needed mostly because the capacitance of normally available capacitors in schools is quite low (millifarads at most) and, as a consequence of that, the transient response lasts a very short period of time during charging and discharging the capacitor (the duration of the transient response is directly proportional to the capacitor capacitance C and to the resistance of the connected resistor R). This has certain disadvantages, lets mention two of them. For example, it's not possible to sensibly carry out an experiment with a light bulb connected to charging circuit of a capacitor and show to the pupils how the light bulb glows at the beginning of the discharge and that the glow gradually decreases as the source voltage approaches the capacitor voltage. Likewise, it is impossible to illustratively demonstrate the dependence of the transient response duration on the magnitude of the resistor resistance. These factors can have a negative influence on the pupils' grasp as they don't have the possibility of first-hand contact with the functioning of the capacitors. Actually, nowadays relatively simple and cheap solution exists – the use of so-called **supercapacitors** which have a very high capacitance.

Supercapacitors and Their Characteristics

It's generally well-known that the capacitance of a parallel-plate capacitor is directly proportional to the plate areas and to the dielectric constant (relative permittivity) of a dielectric filled inside, and is indirectly proportional to the distance of the plates. What are the options if we want to significantly increase the capacitance of the component? A dielectric of higher quality won't have a greater impact, because it's very difficult to find a material with a dielectric constant greater than 10 (e.g. hafnium oxide whose preparation in the form of a thin layer is quite demanding). Therefore, it remains to extensively enlarge the plates' area or shorten the distance between them. However, the enlargement of the plate area mustn't go against the effort to miniaturize the relevant components. This brings us to the world of nanotechnology, where in recent years there has been such a significant development that capacitors with a corresponding plates distance in the order of tenths of a nanometre are already commonly manufactured (it is actually a bilayer thickness at the electrode-electrolyte interface in which energy is concentrated – *Supercapacitor* on Wikipedia) and a specific electrode area of up to 3 000 square meters per gram. Thanks to this, it is possible to obtain supercapacitors with a capacity of thousands of farads.

Carbon nanotubes, for example, appear to be a promising material for electrodes, and graphene is also often used. However, there is a considerable amount of production technologies and usable materials. This issue is summarized in detail, for example, in a study [1]. The price of supercapacitors, which are available in e-shops, usually significantly increases with the capacitance. While a 1.5 F capacitor can be purchased for less than CZK 100, a capacitance of 7.5 F will cost more than CZK 200, a 500 F capacitor will cost approximately CZK 800, and an electrolytic supercapacitor with a capacity of 3 000 F will cost more than CZK 3 000.

As the capacitance of supercapacitors increases, the maximum allowable voltage U decreases which is essential for the maximum energy that the capacitor can store. The reason is simple – the increase of a capacitance is achieved by a significant shortening of the distance between the electrodes d, which in consideration of the relationship for the intensity of the electric field E inside the plate shaped capacitor $E = \frac{U}{d}$ leads to very high intensity even at low voltage. Therefore, there is a risk of dielectric breakdown and destruction of the capacitor. In the case of components with a capacitance in the order of hundreds or thousands of farads, the maximum allowable voltage is usually only about 3 V. Energy E_0, which can be stored in a capacitor, is still huge, for example, if we have the capacitance C = 3 000 F and the voltage U = 2.7 V, we get the energy $E_0 = \frac{1}{2} \cdot C \cdot U = 10\ 935$ J. It's not then surprising, that supercapacitors are used mainly in applications where it is necessary to store a large amount of energy. The importance of these components for electro-mobility is often discussed, where they could replace batteries (the advantage of capacitors is fast charging, but low energy density is problematic). More information about applications can be found, for example, in [2].

The Use of Supercapacitors in Teaching

The fundamental question is how to didactically transform this dynamically developing field into school physics. It is of course possible to mention supercapacitors as an interesting alternative to batteries and focus on their applications, including a discussion of advantages and disadvantages. To compute exercises inspired by the real characteristics of these components may also be interesting. Special attention should be paid to understanding why large capacitance leads to very small maximum voltages so that pupils realize why capacitors have a voltage indication next to its capacitance (for example, it is possible to calculate the intensity of the electric field for given values and compare it with the dielectric strength of different materials, etc.).

In our opinion, the most appropriate is to use these components to directly demonstrate the characteristics of capacitors and for the experimental work of pupils. For this purpose, it is absolutely unnecessary to spend large sums of money on supercapacitors with extremely large capacitance, values of capacitance in the order of farad units are sufficient, where the price is far from large and there is therefore a better chance to order more pieces. In the next section, we will present a few ideas for specific activities in this area.

Charge and discharge curve of a capacitor

With a supercapacitor and a potentiometer (or a resistance decade), it is easy to demonstrate how the charging speed depends on the resistance in the circuit. Connect a voltmeter to the capacitor that is being charged and observe an increasing voltage on the capacitor. If we increase the resistance abruptly, the charging speed will significantly slow down. With an appropriate choice of the resistance, the transient response will be so slow that pupils can note down values of the voltage after certain time steps and then process obtained data in Excel and find out to what extent they are in accordance with the exponential course of the voltage. It is also possible to connect a light bulb into the charging circuit and observe how the glow gradually decreases as the capacitor voltage increases. It is possible to proceed similarly when the capacitor discharges, where it is interesting to observe that the capacitor slowly discharges itself after disconnecting the source, even without connecting a resistor. This happens due to the so-called *leakage current* that passes through the capacitor plates, because the dielectric is not a perfect insulator. The magnitude of this leakage current (and therefore the speed of self-discharge) is an important characteristic of the quality of the component. This effect is noticeable with commonly available supercapacitors, but it is not so strong that its speed could compete with the discharge through the resistor.

Paradox of two capacitors

This is a very interesting problem when a discharged capacitor is connected to an identic capacitor with capacity C charged to voltage U by charge Q. As a consequence of the fact that the voltages on both capacitors approach the same value and due to the charge preservation law and the well-known formula $Q = C \cdot U$, the charge $\frac{Q}{2}$ and the voltage $\frac{U}{2}$ will be the same on both capacitors. Before connecting of an uncharged capacitor into the circuit, the energy of electric field was defined by the formula $E_A = \frac{Q^2}{2 \cdot C}$. After connecting the capacitor into the circuit, the sum of energies of the electric fields in both capacitors is given by the formula $E_B = E_1 + E_2 = \frac{\left(\frac{Q}{2}\right)^2}{2 \cdot C} + \frac{\left(\frac{Q}{2}\right)^2}{2 \cdot C} = = \frac{Q^2}{4 \cdot C} = \frac{1}{2} \cdot E_A$.

So we can see that the total energy of the electric field has decreased to half after connecting an uncharged capacitor into the circuit. The question is what has happened to the other half of the energy? If we do not connect a resistor between a charged and an uncharged capacitor and simply connect them together by conductors, only few of us will think that the other half of the energy has changed into heat in these conductors. Actually, this is the correct answer, which can be proved with the help of the Joule-Lenz's law and higher mathematics (the point is to show that the heat released is independent on the resistance and is always equal to the half of the energy of the electric field in the capacitor). The proof is given in [3], for more detailed problem solution from the didactic point of view containing references to another relevant literature please see [4].

What is really important is how to experimentally demonstrate this paradox. As an interesting demonstration appears to be the connection of a charged and an uncharged capacitor through an object with low resistance, for example through a strip of an aluminum foil. Considering that the time constant of the transient response is directly proportional to the resistance (and therefore is very small), the current will be so huge to burn the aluminum strip, which obviously demonstrates the considerable amount of accumulated energy stored in the capacitor (for example, for the capacitance of 7.5 F and the voltage of 5 V, the energy is almost 100 J), and then the release of a substantial part of the energy into the conductor connecting the two capacitors, even if its resistance is very small. Demonstration can be done in many ways, for example, it is possible to 'draw' by discharge using the outlets of a sharp conductor into the aluminium foil when discharging the capacitor, etc.

Laboratory work – equilibrium of a circuit with a capacitor

Let us present an idea for a particular laboratory work closely related to the paradox of two capacitors. Consider a circuit with a capacitor connected as shown in Figure 1 below. The notation used in figure 1 will also be used in all the following computations. At the beginning, the capacitor is charged to the source voltage U_0 and the current doesn't flow through the circuit. At time $t = 0$ s, the switch S is closed and the capacitor starts to discharge through the resistor R_2. At the same time, however, the capacitor is still connected to the source of voltage, and therefore it doesn't discharge completely and its voltage gets settled at the value U_u after a certain time. It's necessary to experimentally determine what this steady value of the capacitor voltage will be and for how long the transient response will last, and also compare the measured values with the theory.

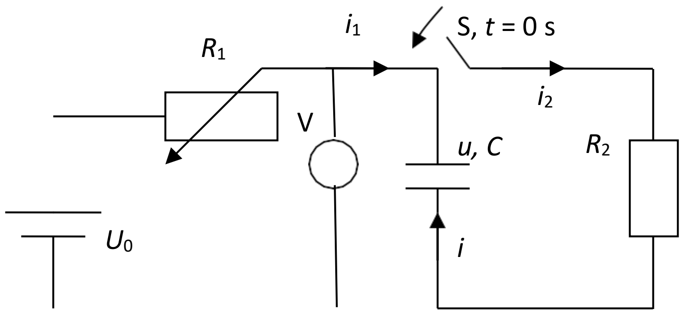


Figure 1. A diagram of a circuit with a capacitor

At steady-state, the capacitor voltage U must be the same as the voltage drop across the resistor R_2, so the equation $U_u = R_2 \cdot i_2$ must apply. At the same time, the voltage division between the source voltage and the capacitor voltage must be equal to the potentiometer voltage drop, so the equation $U_0 - U_u = R_1 \cdot i_1$ must apply. At steady-state, the current i_1 equals to the current i_2 (the capacitor is neither charging or discharging). So it applies:

$$U_0 - U_u = R_1 \cdot i_1 \rightarrow U_0 - U_u = \frac{R_1}{R_2} \cdot R_2 \cdot i_2 \rightarrow U_0 - U_u = \frac{R_1}{R_2} \cdot U_u \rightarrow U_u = \frac{R_2 \cdot U_0}{R_1 + R_2}. \qquad (1)$$

It is more difficult to find the solution of the transient response, which is based on determining the dependence of the capacitor voltage on time $u(t)$. After applying Kirchhoff's laws and the formula $Q = C \cdot U$ and a few modifications, we obtain the differential equation for the unknown function $u(t)$ with the initial condition $u(0) = U_0$ in the form:

$$C \cdot \frac{du}{dt} + u \cdot \frac{R_1 + R_2}{R_1 \cdot R_2} = \frac{U_0}{R_1}\,. \qquad (2)$$

There we have a linear differential equation with the right-hand side, whose particular solution $u_p = \frac{R_2 \cdot U_0}{R_1 + R_2}$ obviously corresponds to the steady-state of the circuit. After solving the homogeneous differential equation with regard to the initial condition, we obtain the time formula:

$$u(t) = \frac{R_1 \cdot U_0}{R_1 + R_2} \cdot e^{-\frac{R_1 + R_2}{R_1 \cdot R_2 \cdot C} \cdot t} + \frac{R_2 \cdot U_0}{R_1 + R_2}\,. \qquad (3)$$

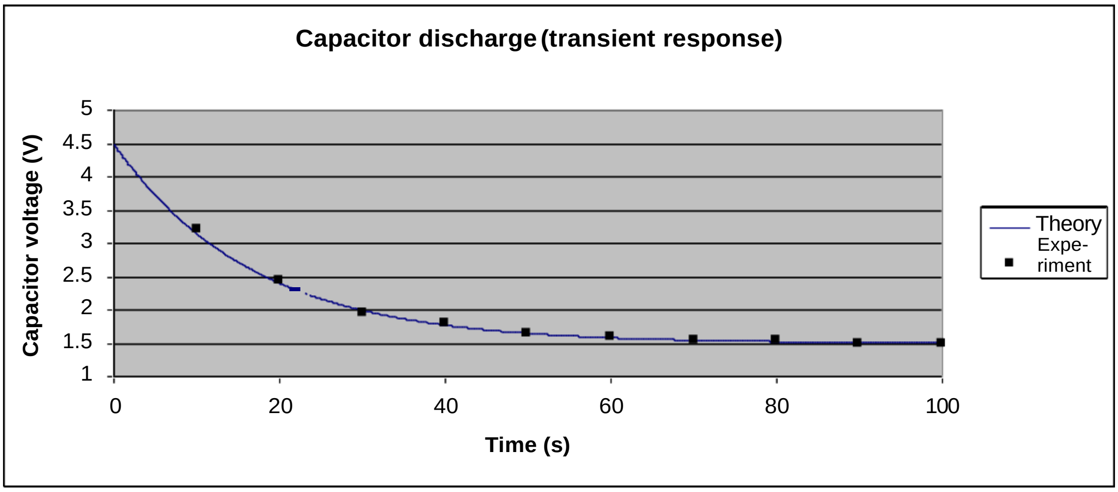
The first component corresponds to the transient response, while the second component corresponds to the steady-state of the circuit. The time duration of the transient response is given by the formula

$$\tau = \frac{R_1 \cdot R_2 \cdot C}{R_1 + R_2}\,, \qquad (4)$$

indicating the **time constant** of the process. Therefore, it is obvious that the time duration extends with increasing capacitor capacity and is dependent on the resistor resistance and the potentiometer resistance.

To perform the experimental task, we need a voltage source (flat battery), a capacitor of a large capacity (e.g. 1.5 F), a digital voltmeter, a resistor with a resistance in the order of ohms, a potentiometer or a decade resistance, a digital multimeter, a stopwatch, a switch and connecting conductors. We connect the circuit according to the Figure 1 and let the capacitor be charged through a potentiometer, for which we measure the resistance with a digital multimeter. During charging, we observe how the resistance changes of the potentiometer affect the charging speed of the capacitor, whose voltage is measured with a voltmeter. After the voltage value stabilizes, we try to disconnect the voltage source for a while and observe how fast the capacitor discharges if the appliance is not connected. We note the voltage drop within one minute and then recharge the capacitor to the maximum value by reconnecting it to the source. After that, we turn on the switch and the stopwatch and observe the voltage drop through the capacitor (the potentiometer is set to about 10 ohms). Regularly after about 10 seconds, we note the valid voltage value as the value stabilizes. We choose the stabilization criterion; it can be, for example, that the digital voltmeter reading does not change for at least 5 seconds. We write down (in addition to the resistance value set on the potentiometer) the steady voltage value and the transient response duration time. Then we turn off the switch, charge the capacitor to the maximum again and repeat the process for a different resistance value of potentiometer R_1. This time, however, we no longer note consecutive voltage values, but only determine the time duration of the transient response and the resultant voltage.

As a part of the data processing, pupils should find out how the time duration of the transient response depends on the potentiometer resistance R_1. The data obtained can be plotted into a graph. Furthermore, pupils should be able to compare the theoretical and real course of a voltage drop across the capacitor and also compare the measured steady-state value with the theory, for example, according to the example given in the Graph 1 (values $U = 4.5$ V, $R_1 = 10\ \Omega$, $R_2 = 5\ \Omega$, $C = 1.5$ F).



Graph 1. Transient response – The capacitor is partially discharging

As a part of the protocol elaboration, the questions related to, for example, the self-discharge of the capacitor, the limited allowable voltage and the causes of the differences between the theoretical and experimental course should be answered. It is obvious that pupils can hardly master the above mentioned derivation using a differential equation. However, they should understand the basic formulas leading to its compilation and also be able to analyze the above mentioned formulas (3) and (4).

Conclusion

The paper presents an introduction to the use of supercapacitors in upper secondary school physics instruction. It outlines some possibilities on how to use these interesting components in order to increase pupils' motivation and improve their grasp of the capacitors function. It can be assumed that with the further development of this technique, it will become easier to purchase supercapacitors for schools in sufficient quantities and thus broaden the offer of laboratory works for pupils.

References

[1] Sharma, K. a kol. Review of supercapacitors: Materials and Devices. *Journal of Energy Storage*, 2019, 21: 801-825.

[2] Libich, J. a kol. Supercapacitors: Properties and applications. *Journal of Energy Storage*, 2018, 17: 224-227.

[3] Kohout J. Řešené příklady z elektřiny a magnetismu, úloha 175 [Solved problems from electricity and magnetism, task number 175]. Available from: http://home.zcu.cz/~jkohout4/EMG.doc [cit. 15.8.2019].

[4] Biolek Z.: Problém dvou kapacitorů v pedagogické praxi [The problem of two capacitors in pedagogical practise]. Available from: http://www.valachnet.cz/biolek/articles/sto10.pdf [cit. 15.8.2019].